Reachable Sets Analysis—An Efficient Technique for Performing Missile/Sensor Tradeoff Studies

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The standard method of performing tradeoffs between missile controllability and sensor accuracy in a one-on-one intercept problem is to develop a Monte Carlo guidance simulation and determine the effect on average miss-distance of varying the system parameters. This paper describes a new and more efficient method—reachable sets analysis. This form of analysis is concerned with the evolving sets of states that may be reached at future times by the target and interceptor (missile) vehicles. The method determines directly whether there is adequate information available about the target state at each time to allow an intercept to be made. It is shown that if certain conditions on the reachable sets of the two vehicles are met there exists a guidance law that will always achieve the required miss-distance in real time; if the conditions are not met there is no such guidance law. As an example, the paper briefly describes how reachable sets techniques can be used to perform missile/sensor tradeoffs in the context of homing guidance.

Introduction

REACHABLE sets analysis is concerned with the properties of the sets of reachable states of a system. Conceptually, almost any control or estimation problem can be treated by reachable sets analysis[1,2] once the criteria of system performance have been defined in terms of desired properties of the reachable sets. In practice, reachable sets analysis has had most impact as a theoretical tool in studying abstract properties of systems; it has had little impact as a numerical analysis tool. This is mainly because it is difficult to describe complex n-dimensional sets which evolve and change as time proceeds. Bounding ellipsoids and bounding polyhedra have been introduced respectively in Refs. 2 and 3 to alleviate this difficulty. However, for a complex nonlinear system, a troublesome tradeoff usually remains between the accuracy of approximating the true reachable sets and the complexity of the numerical calculations. It is simpler to treat such a system by single-trajectory techniques, such as linearizing about a nominal trajectory, which leads to extended Kalman filtering in the case of estimation, or to gradient techniques in the case of optimization. Some idea of the properties of the sets of possible system states can then be obtained by performing many single-trajectory experiments. However, when reachable sets analysis is numerically feasible it can be very rewarding, both in providing improved understanding of the system under study, and in efficiency—providing with one pass a complete analysis of all possible trajectories of the system. This paper describes techniques for applying reachable sets analysis to one-on-one intercept problems—in particular for performing tradeoffs between missile controllability and sensor accuracy.

Formulation

The problem of intercepting a nonmaneuvering vehicle (the target) with a missile (the interceptor) can be formulated as a nonlinear stochastic control problem. The dynamics of the vehicles are described by the state equations

$$\begin{aligned} \dot{x}_I &= f_I(x_I, u, t, p) + w_I, P[x_I(t_o)] \quad \text{known} \\ \dot{x}_T &= f_T(x_T, t, p) + w_T, P[x_T(t_o)] \quad \text{known} \end{aligned} \tag{1}$$

where x_I, x_T are the interceptor and target state vectors, u is the interceptor control vector, $u \varepsilon U(p, x_I)$, p is a vector of system parameters, $U(p, x_I)$ is the set of allowable interceptor controls, t represents time, t_o is the initial time, w_T, w_I are noise processes, and $P[\cdot]$ represents an a priori probability density.

The noise terms w_T and w_I are introduced to approximate the actual effects of uncertainties in the vehicle dynamic equations. The process by which information is gathered about the vehicle states is described by the measurement equations

$$z = h(x_T, x_I, p, t) + v \tag{2}$$

where z is the measurement vector and v is a measurement noise process.

It is assumed that the first three elements of x_I and x_T are the three-dimensional Cartesian position vectors of the interceptor and target. These subvectors are denoted respectively by y_I and y_T.

The parameter vector p includes any parameters describing the capabilities of the interceptor, the target, and the measurement process, which the system designer may wish to trade off. Examples are: 1) limits on missile lateral and/or axial control accelerations, 2) missile delays in response to commands, 3) missile thrust parameters, 4) missile aerodynamic properties, 5) missile launch dispersions, 6) target dynamic properties, 7) single measurement sensor error, 8) target acquisition range, and 9) target track rate. It is assumed that a successful intercept is achieved if the true positions of the interceptor and the target are brought within a distance D at some time near the nominal terminal (intercept) time t_f.

To perform missile/sensor design tradeoffs it is desired to determine various combinations of the parameters p, that allow a successful intercept to be made with specified probability. The reachable sets method treats this problem by considering the sets of positions that the target and missile may reach at times near the nominal terminal time t_f. The following sections discuss, in turn, the target reachable positions, the interceptor reachable positions, and the method for determining if the missile reachable positions are sufficiently large in relation to those of the target.

Target Uncertainty Volume

It is assumed that a target tracking algorithm is employed to operate on the sensor data and generate an estimate $\hat{x}_T(t|t)$,

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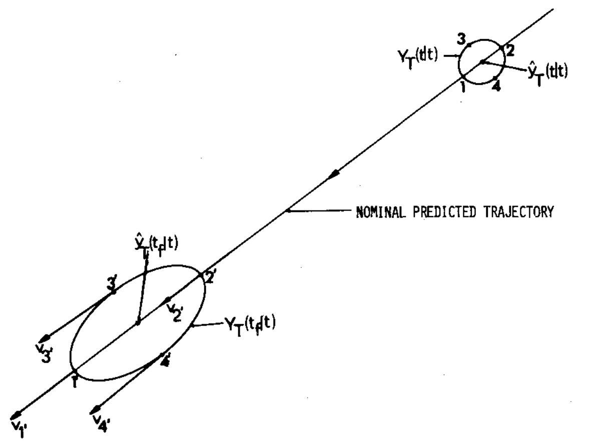

Fig. 1 Predicted target uncertainty volume.

and a covariance matrix $P(t \mid t)$, of the target state at time t based on measurements up to time t. It is necessary to determine the sets of positions the target may reach at times near t_f on the basis of the measurements to the present time t. Normally, it is sufficiently accurate to determine these sets of positions from the set of positions reachable at exactly time t_f and from the associated velocities. This information is included in the set $X(t_f \mid t)$ of target states reachable at t_f given measurements to time t.

If the target dynamics are linear and the statistics gaussian, it is straightforward to employ standard theory[4] and determine on the basis of measurements to time $t, \hat{x}_T(t_f \mid t)$ and $P(t_f \mid t)$—the target state estimate and its covariance at time t_f. The three-sigma uncertainty volume of reachable target states is then given by[4]

$$X_T(t_f \mid t) \equiv \{x_T(t_f) \mid [x_T'(t_f) - \hat{x}_T'(t_f)]\, P^{-1}(t_f \mid t)[x_T(t_f) - \hat{x}_T(t_f)] \leqq 3^2\} \tag{3}$$

where $\hat{x}_T(t_f \mid t)$ has been abbreviated to $\hat{x}_T(t_f)$. The corresponding three-sigma set $Y_T(t_f \mid t)$ of target reachable positions is given by

$$Y_T(t_f \mid t) \equiv \{y_T(t_f) \mid [y_T'(t_f) - \hat{y}_T'(t_f)]\, Q^{-1}(t_f \mid t)[y_T(t_f) - \hat{y}_T(t_f)] \leqq 3^2\} \tag{4}$$

where $Q(t_f \mid t)$ is the appropriate submatrix of $P(t_f \mid t)$ and $\hat{y}_T$ is the estimate of y_T based on measurements to t.

If the target dynamics are nonlinear or the statistics are non-gaussian, the reachable sets method can still be applied if it is possible to obtain the extrapolated set $X_T(t_f \mid t)$. If the extrapolation is for a short time, it may be sufficiently accurate to apply the linear theory by linearizing the target dynamics about a nominal trajectory. For severely nonlinear dynamics and a long extrapolation time, it may be necessary to employ a process such as the one illustrated in Fig. 1, where several integrations of the nonlinear target dynamic equations are made. For example, an advanced position and velocity, point 1′ in Fig. 1, at time t_f is found by starting from an advanced initial position, point 1, a high initial velocity, and integrating forward using the fastest dynamics that are reasonable on the basis of information to time t. Once a number of bounding positions have been obtained at time t_f, it is often convenient and sufficiently accurate to fit an ellipsoid to them in order to describe compactly the position uncertainty volume.

Interceptor Reachable Positions

Associated with each initial state of the interceptor is a set of positions that are reachable at a given future time. The bounds of the reachable set are determined by the limits of the allowable interceptor controls. The set of interceptor reachable positions at time t_f is denoted by $R[t_f \mid x_I(t)]$. Figure 2 illustrates the projection on a plane of this set for an interceptor with lateral control only. As shown in the figure, different velocities are associated with different boundary points of the set of reachable positions. Note that the set of reachable positions at time t_f for an interceptor with only lateral control has some thickness in the axial direction. This is because the interceptor may cover a shorter distance by making a zig-zag maneuver. If the interceptor has axial control (by some form of control over thrust or drag) the positions reachable at t_f expand in the axial direction.

If the acceptable miss-distance is D then the effective size of the set of interceptor reachable positions is increased in each direction by D. The set of effective reachable interceptor positions is denoted by $R[t_f \mid x_I(t), D]$. Figure 2 also shows the projection on a plane of this set.

In many applications real time uncertainty exists in the interceptor state, $x_I(t)$. This uncertainty may be because of imperfect information about the interceptor state after launch. If uncertainties of this type do exist they must be taken into account by identifying one or more sets $X_I(t_f \mid t)$ of interceptor states that the interceptor can reach with specified probability at time t_f on the basis of information to time t. The corresponding sets of reachable positions at times near t_f can be determined from $X_I(t_f \mid t)$.

Determination of Target Coverage

As discussed in the previous two sections, it is possible to determine an uncertainty volume of possible future positions for the target, $Y_T(t_f \mid t)$, a set of positions the interceptor can reach at future times from a given initial state, $R[t_f \mid x_I(t)]$, as well as the velocities associated with both of these sets of positions. The question to be answered now is whether all the target trajectories corresponding to points in $Y_T(t_f \mid t)$ can be satisfactorily intercepted at some time by an interceptor trajectory corresponding to a point in $R[t_f \mid x_I(t)]$. In other words, does the interceptor reachable set "cover" the target uncertainty volume? The following definitions are introduced to make the notion of coverage more precise. 1) A target trajectory $x_T(\tau)$, $t_o < \tau < \infty$, is reachable from a given interceptor state $x_I(t)$ at time t if an interceptor control $u(\sigma)\varepsilon U(p, x_I)$, $t < \sigma < \infty$ exists which will at some future time t_f bring the interceptor position within a distance D of the target position. 2) The coverage, $C(t)$, is the probability at time t that the target trajectory is reachable, on the basis of statistics of the target state $x_T(t)$ at time t and the statistics of the interceptor state $x_I(t)$ at time t. That is, if W is the event

$$W(t) = \{x_T(\tau), t \leqq \tau < \infty, \quad \text{is reachable from } x_I(t)\}$$

then

$$C(t) \triangleq P[W(t)] \tag{5}$$

Evaluation of the probability in Eq. (5) is based on the statistics of both the target and the interceptor states at time t.

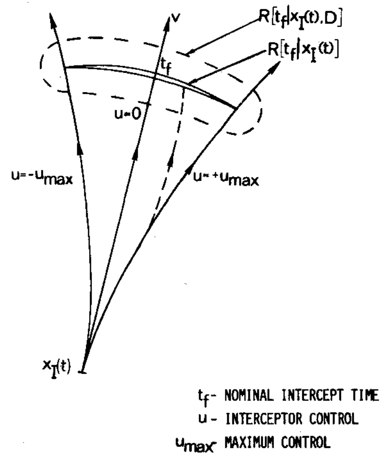

Fig. 2 Interceptor reachable positions for lateral control.

To perform missile/sensor tradeoffs it is necessary to determine critical values of the vector of parameters, p, such that the interceptor is just capable of achieving full coverage over the entire intercept engagement. For practical purposes, it is usually permissible to make two assumptions that simplify the numerical calculation of coverage.

1) The target state uncertainty volume $X_T(t_f|t)$, and the interceptor reachable states $X_I[t_f|t]$ are defined by bounds at a desired sigma level on the statistical variables.

2) For a particular geometry of intercept, suppose the earliest possible intercept could occur at $t_f - \Delta t_1$ where t_f is the nominal time of intercept (i.e., the time at which the nominal interceptor trajectory intercepts the mean predicted target trajectory), and the latest possible intercept occurs at $t_f + \Delta t_2$. Then it is assumed for each target trajectory and each interceptor trajectory that its velocity magnitude and direction are constant over each of the intervals $[t_f - \Delta t_1, t_f]$ and $[t_f, t_f + \Delta t_2]$.

The first assumption enables determination of specific sets of reachable positions which contain the true state with specified probability. If desired, probabilistic information can be obtained by repeating the analysis which follows using sets of states defined at different sigma levels. The second assumption, which is not as strong as assuming that the shapes of the reachable position sets remain fixed near t_f, specifies that on each target and each interceptor trajectory the velocity is constant over each of the time intervals shortly before and shortly after time t_f. This means that the velocity of each interceptor trajectory relative to each target trajectory is also constant over these time intervals. Accordingly, it is possible to fix the set of reachable positions for either the target or the interceptor at those positions attained at the nominal time t_f, and to determine relative motion of the other set of positions by means of the relative velocities at t_f. Figure 3 illustrates this process. It shows the projection on a plane of the target position uncertainty volume, $Y_T(t_f|t)$, at the nominal intercept time t_f based on measurements to time t. Also shown is the set, $R[t_f|x_I(t)]$, of positions the interceptor can reach at t_f from a given initial state $x_I(t)$. The interceptor has both lateral and axial control. The figure shows a snapshot of the reachable positions at the particular time t_f. As time proceeds, the sets of reachable positions travel and also alter in size and shape. The effect of these movements can be taken into account, given assumption 2 above by holding the target uncertainty volume $Y_T(t_f|t)$ fixed in position, and moving the interceptor reachable positions in the direction of the interceptor velocity relative to the target. The motion of point 1 on $R[t_f|x_I(t)]$ relative to point A on the boundary of $Y_T(t_f|t)$ is shown as the vector v_{R_1}. The critical situation of just achieving reachability occurs when, as shown in Fig. 3, the relative velocity vector is tangential to the target uncertainty volume. A second critical point is shown in Fig. 3 where the target trajectory corresponding to point B on the boundary of $Y_T(t_f|t)$ is just reachable by the interceptor trajectory corresponding to point 3 on $R[t_f|x_I(t)]$. Thus, Fig. 3 illustrates, in two spatial dimensions, the critical situation where, on the basis of information to the particular time instant t, the interceptor, target, and measurement parameters p, are just adequate to achieve full coverage.

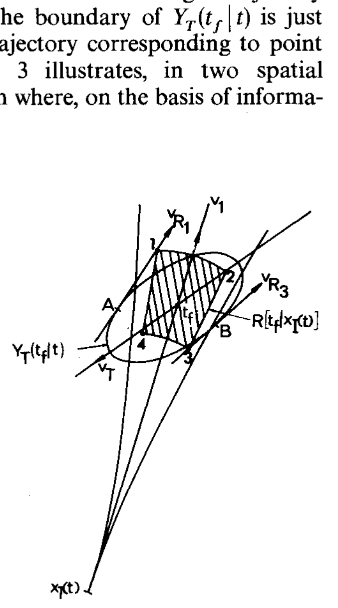

Fig. 3 Critical geometry for full target coverage (for interceptor with lateral and axial control).

Having full coverage at a particular time instant does not, of course, guarantee coverage at all times within the interval $[t_o, t_f]$. Therefore, it is necessary to check coverage at various times within this interval.

As a result of the requirement for checking coverage from every possible initial interceptor state within the set $X_I[t|t_o]$ at all times t in the interval $[t_o, t_f]$ it would appear that reachable sets analysis requires an enormous investment in computer trajectory computation as well as in graphical analysis. However, in practice the contrary is true: the method is extremely efficient in enabling study of a wide range of system parameters at modest computational cost. There are two principal reasons for this.

1) As implied in Fig. 3, only a very small number of interceptor trajectories (usually two trajectories, both lying in a single plane) are critical in determining coverage. The trajectories of concern in any particular case can often be found simply by observing the shape of the reachable sets.

2) Usually, the target uncertainty volume of positions shrinks at a different rate than does the set of interceptor reachable positions. Thus, an accurate check of the coverage at a single critical time is often all that is needed after some initial experience with the problem.

Relationships between Reachable Sets Analysis and Real Time Guidance

Two properties of the reachable sets make clear the relationship between conditions on the reachable sets and real time guidance.

Property 1

If full coverage is not maintained from t_o to t_f, i.e., if $C(t) < 1.0$ for some $t \varepsilon [t_o, t_f]$, there is no guidance law that can guarantee a successful intercept. In fact the probability p_k of achieving a successful intercept is bounded by the minimum value of the coverage, i.e.,

$$p_k \leqq \min_{t \varepsilon [t_o, t_f]} C(t)$$

Property 2

If full coverage is maintained from t_o to t_f, i.e., $C(t) = 1.0$ for all $t \varepsilon [t_o, t_f]$, a real time guidance law exists that can guarantee a successful intercept, i.e., a real time guidance law exists for which $p_k = 1.0$.

The first property follows directly from the definitions of coverage and reachability. For example, a coverage of say 0.8 at some time implies that with probability 0.2 there is no intercept control bringing the interceptor within a distance D of the target at any future time. Thus, no real time intercept law could guarantee a probability of successful intercept greater than 0.8.

The second property follows from the definitions of coverage and reachability and the controllability of the interceptor reachable positions. By applying appropriate controls, the interceptor reachable positions at a time t_2 can be made to shrink in any desired position within the reachable positions at an earlier time t_1. Thus, a "reachable sets guidance law" can be proposed: "control the position of the shrinking interceptor reachable positions so that coverage is always maximized." If this reachable sets guidance law is used, a successful intercept is guaranteed, given that full coverage is maintained from t_o to t_f.

Numerical Results for a Homing Guidance Example

The ideas presented above have wide application in one-on-one intercept problems. In practice the accuracy of the analysis

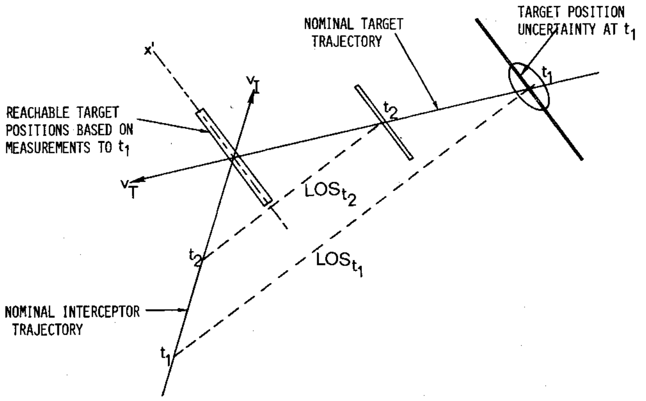

Fig. 4 Example geometry of a homing engagement.

will depend of course on the accuracy of modeling the sensors and the vehicle dynamics, and the accuracy of computing the reachable sets of positions. The following homing guidance example using simple models for the vehicles and sensors is intended both to illustrate how the method is applied and how improved understanding of an intercept problem can be gained from a modest investment in reachable sets analysis.

Figure 4 illustrates an example of homing guidance intercept geometry. Nominal trajectories for the target and interceptor are shown with the line-of-sight direction from the interceptor to the target shown at times t_1 and t_2. The ellipse drawn about the nominal target positions at time t_1 illustrates the initial uncertainty about the target position. The superimposed rectangle represents the single-measurement accuracy of a sensor‡ onboard the interceptor. The intersection of the rectangle and the ellipse represents the uncertainty about the target position immediately after acquisition of the target by the onboard sensor. The rectangles drawn about the predicted target trajectory at t_2 and at the nominal intercept point represent the uncertainty as to where the target will be at those future times, on the basis only of the information available at time t_1, immediately after target acquisition. Note that the line-of-sight direction remains relatively constant so that while information received at later times, such as t_2, reduces the size of the target uncertainty volume extrapolated to the nominal terminal time, the greatest uncertainty as to where the target will be remains in the x' direction shown in Fig. 4.

Figure 5 illustrates for the same example geometry the effective reachable positions of an interceptor with lateral control and a small warhead. The limits of reachability achieved by these

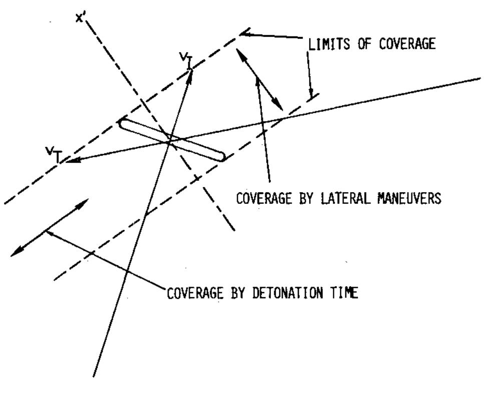

Fig. 5 Limits of interceptor coverage.

‡ It is assumed that the sensor is relatively accurate in range measurements and relatively inaccurate in angle measurements.

effective reachable positions are shown by the dashed lines drawn in the direction of the interceptor velocity relative to the target velocity. The orthogonal directions in which coverage is achieved 1) by precise timing of the warhead detonation, and 2) by lateral maneuvers of the interceptor, are also shown. Note that the x' direction, the direction of greatest uncertainty about the target position (assuming a sensor with relatively inaccurate angle measurements) is also the direction in which coverage is achieved only by lateral maneuvers of the interceptor.

Because obtaining coverage in the x' direction is the primary factor determining requirements on the sensor angle accuracy and on the interceptor lateral acceleration, an analysis of the coverage in the x' direction is of particular interest. To simplify the analysis, the following assumptions are made: 1) The line-of-sight direction is assumed to be constant (so that the contribution of range measurements to reduction of the x' uncertainty is neglected) and 2) the target position in the x' direction is governed by the equation

$$x'(t) = x_f - v_f t$$

where the scalars x_f, v_f are the position and velocity at time t_f, and t is the time-to-go (so unknown accelerations are neglected).

With these assumptions, a line-of-sight measurement made at time-to-go, t_i, provides an estimate z_i of the true position, $x_f - v_f t_i$, in the x' direction. That is

$$z_i = x_f - v_f t_i + \varepsilon_i \tag{6}$$

where ε_i is a noise term with zero mean and variance $R_i^2 \sigma_\theta^2$, $R_i = \hat{v} t_i$ is the target-interceptor range at t_i, $\hat{v}$ is the closing velocity, and σ_θ is the one-sigma angle accuracy of the LOS sensor. If N independent measurements are made at times $t_1, \ldots, t_N$, Eq. (6) can be written in matrix form

$$z = Hx + \varepsilon \tag{7}$$

where

$$x = \begin{bmatrix} x_f \\ v_f \end{bmatrix}$$

$$z' = [z_1, \ldots, z_N]$$

$$\varepsilon' = [\varepsilon_1, \ldots, \varepsilon_N]$$

H is an $Nx2$ matrix

Using standard theory, the covariance matrix, P, of x is given by

$$P^{-1} = H'R^{-1}H \tag{8}$$

where R is the covariance of ε, ($R = E[\varepsilon\varepsilon^T]$). From Eq. (8) it can be shown that the standard deviation of the estimated target position x_f at time t_f, based on measurements made at times $t_1, t_2, \ldots, t_N$, is

$$\sigma_{x'} = \frac{\hat{v}\sigma_\theta}{\left[\sum_{i=1}^{N} \frac{1}{t_i^2} - \frac{1}{N}\left(\sum_{i=1}^{N} \frac{1}{t_i}\right)^2\right]^{1/2}} \tag{9}$$

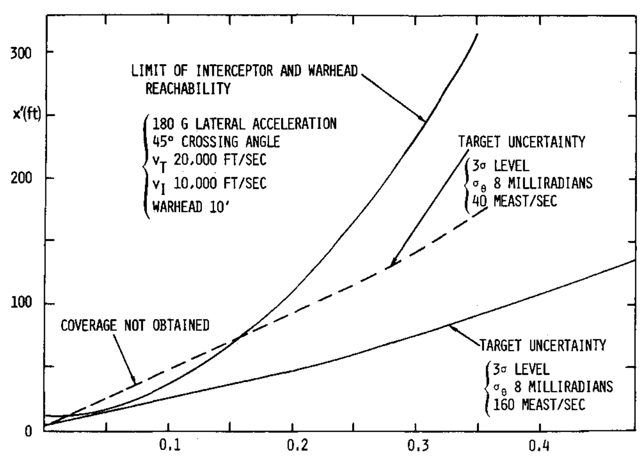

Fig. 6 Coverage of the interceptor and warhead vs time-to-go.

Figure 6 shows the limit of reachability in the x' direction of an example interceptor and warhead as well as two cases of uncertainty at the three-sigma level about the target positions in the x' direction [obtained from Eq. (9)]. In one case, (when the measurement rate is 40 measurements/sec) coverage is not maintained from about 0.16 sec to go until about 0.02 sec to go. In this case the sensor accuracy and the interceptor controllability are inadequate to ensure the success of an intercept. In the other case (when the measurement rate is increased to 160 measurements/sec) coverage is just maintained at all times-to-go. Note that the smallest margin of coverage occurs at about 0.05 sec to go, implying that a superior guidance law and estimation algorithm are needed in the last fractions of a second before intercept. Conversely, there is an adequate margin of coverage until about 0.2 sec-to-go implying that less accurate guidance and estimation algorithms would be acceptable up to this time.

A more accurate analysis of the problem requires the development of an estimation error analysis corresponding to an optimal estimation algorithm capable of incorporating both range and angle measurements. Also the use of deliberate interceptor maneuvers to modify the line-of-sight direction could be investigated by determining how the reachable sets are affected.

Conclusions

Reachable sets analysis is a powerful tool for analyzing a system whenever it is possible 1) to identify and numerically describe the reachable sets of the system and 2) to express the criteria of system performance in terms of conditions on the sets of states. One application where these conditions are usually met is that of performing missile/sensor tradeoffs during the design of an intercept system. In this application, reachable sets analysis provides an efficient method of studying the implications to the system of the many possible variations of sensor and interceptor parameters. The analysis is independent of the guidance law showing what it is possible to achieve with a perfect guidance law—not what is achieved with a given guidance law that may or may not be optimal. Often the method provides increased understanding of the precise difficulty that may be preventing an intercept, which allows the designer to make appropriate modifications.

While reachable sets analysis is much more efficient than is simulation for performing initial missile/sensor tradeoffs, it does not entirely supplant simulation, which is valuable for verification of the chosen parameters and for verification of the chosen real time guidance law. The guidance law should be modified in the simulation phase of the system design until it yields results close to the achievable results predicted by the reachable sets analysis.

In addition to the homing guidance application described in this paper applications of reachable sets analysis have been made in the areas of missile/radar/intercept geometry tradeoffs in command guidance for intercepting a re-entry vehicle, determination of minimum target track rates, and determination of the "footprint" of sites that are defendable vs a given target by a command guidance radar/interceptor combination.

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